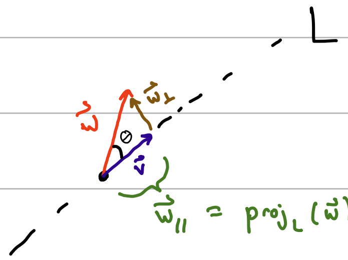
Lecture 22

· We are talking about sections 5.1/5.2   Orthogonality + Subspaces



$$\vec{w}_{\parallel} = \text{proj}_L(\vec{w}) = \frac{\vec{w}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\,\vec{v}$$

- $\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}$
  - $\vec{w} = \vec{w}_{\parallel} + \vec{w}_\perp$
  - $\|\vec{w}\|^2 = \|\vec{w}_{\parallel}\|^2 + \|\vec{w}_\perp\|^2 \geq \|\vec{w}_{\parallel}\|^2$

$$\rightarrow \vec{w}\cdot\vec{w} \geq \left\| \frac{\vec{w}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\,\vec{v} \right\|^2 = \left(\frac{\vec{w}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)^2 \vec{v}\cdot\vec{v} = \frac{(\vec{w}\cdot\vec{v})^2}{\vec{v}\cdot\vec{v}}$$

$$\rightarrow (\vec{v}\cdot\vec{v})(\vec{w}\cdot\vec{w}) \geq (\vec{w}\cdot\vec{v})^2$$

$$\rightarrow \|\vec{v}\|\,\|\vec{w}\| \geq |\vec{w}\cdot\vec{v}|$$

<u>Def</u> – A set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\} \subset \mathbb{R}^n$ is <u>orthogonal</u> if $\vec{v}_i \cdot \vec{v}_j = 0 \quad \forall\, i \neq j$

example: $\{\vec{e}_1, \dots, \vec{e}_n\} \subset \mathbb{R}^n$ is $\perp$

- In fact, this set is <u>orthonormal</u>, i.e. we also have that all its vectors are unit vectors

$\boxed{eg}$ $\vec{v} \in \mathbb{R}^n$

$\Rightarrow \{\vec{v}\}$ is $\perp$

$\boxed{eg}$ $\left\{ \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix} \right\} \subset \mathbb{R}^3$ is $\perp$

• **Prop**: An $\perp$ set of nonzero vectors $\{\vec{v}_1, \dots, \vec{v}_k\} \subset \mathbb{R}^n$ is linearly independent.

· **Proof**: Suppose we have a linear relation

$$c_1\vec{v}_1 + \dots + c_k\vec{v}_k = \vec{0}$$

$$\Rightarrow \vec{v}_i\,(c_1\vec{v}_1 + \dots + c_k\vec{v}_k) = \vec{0} \cdot \vec{v}_i$$

$$\Rightarrow c\,\vec{v}_i \cdot \vec{v}_1 + \dots + c_k\,\vec{v}_i \cdot \vec{v}_k = 0 \implies c_i\, v_i \cdot v_i = 0 \Rightarrow c_i = 0$$

$$c_1 = \dots = c_k = 0 \Rightarrow \{\vec{v}_1, \dots, \vec{v}_k\} \text{ is linearly independent}$$

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Theorem: Let $W \subset \mathbb{R}^n$ be a non-trivial subspace. Then W has an orthonormal basis.

· **Proof** (by recipe): Let $\beta = \{v_1, \dots, v_k\}$ be basis for W. I claim that we get a $\perp$ basis $\{\vec{w}_1, \dots, \vec{w}_k\}$ for W as follows:

$$\vec{w}_1 = \vec{v}_1$$

$$\vec{w}_2 = \vec{v}_2 - \text{proj}_{\vec{w}_1}(\vec{v}_2)$$

$$\vec{w}_3 = \vec{v}_3 - \text{proj}_{\vec{w}_1}(\vec{v}_3) - \text{proj}_{\vec{w}_2}(\vec{v}_3)$$

$$\vdots$$

$$\vec{w}_k = \vec{v}_k - \text{proj}_{\vec{w}_1}(\vec{v}_k) - \dots - \text{proj}_{\vec{w}_{k-1}}(\vec{v}_k)$$

$\boxed{\text{Ex}}$ Find an orthonormal basis for the plane $P = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : 2x_1 + x_2 - x_3 = 0 \right\}$

Solution: Have a (not $\perp$) basis. $\left\{ \underset{\vec{v}_1}{\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}}, \underset{\vec{v}_2}{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}} \right\}$

Get an $\perp$ basis $\{\vec{w}_1, \vec{w}_2\}$ by setting

$$\vec{w}_1 = \vec{v}_1$$

$$\vec{w}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1} \vec{w}_1 = \vec{v}_2 - \operatorname{proj}_{\vec{w}_1}(\vec{v}_2)$$

$$= \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - \frac{1}{5} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \frac{1}{5} \left[\begin{pmatrix} 5 \\ 0 \\ 10 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \right] = \frac{1}{5} \begin{pmatrix} 4 \\ 2 \\ 10 \end{pmatrix}$$

So my $\perp$ basis for P is

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \frac{1}{5} \begin{pmatrix} 4 \\ 2 \\ 10 \end{pmatrix} \right\} \Rightarrow \left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \right\} \underset{\text{basis}}{\overset{\text{orthonormal}}{\Rightarrow}} \left\{ \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{30}} \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \right\}$$